Criteria for evaluation of universal formulas for percolation thresholds

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Several universal formulas that predict approximate values for percolation thresholds of all periodic graphs have been proposed in the physics and engineering literature. The existing universal formulas have been found to have substantial errors in their predictions for some lattices. This paper proposes a set of desirable criteria for universal formulas to satisfy, and investigates which criteria are satisfied by two bond threshold formulas and two site threshold formulas most cited in the literature. The analysis is limited to lattices in two dimensions.

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I. INTRODUCTION

Since the origins of percolation theory [1], the determination of percolation thresholds has been a challenging problem. Exact solutions have been found only for arbitrary trees [2] and a small number of periodic two-dimensional graphs [3–6]. For other graphs of interest, the problem has been approached by simulation and estimation, e.g., [7,8], and through rigorous bounds, e.g., [9–18].

Research on percolation thresholds attempts to understand the dependence of the percolation threshold upon the detailed structure of the underlying lattice graph. For over 40 years there have been efforts to find a *universal formula*, based on a small number of features of the underlying lattice, for predicting the values of the percolation threshold for all lattice graphs. We provide a few important examples.

Vyssotsky *et al.* [19] studied bond percolation on eight regular two- and three-dimensional lattices. They commented that "p_c appears to be little affected by differences of lattice type if the number of dimensions and coordination number are the same," and mentioned the approximation

$$p_c = \frac{d}{(d-1)q}, \tag{1}$$

where d is the dimension of the lattice and q is the coordination number (or vertex degree) of the lattice.

For site percolation, the formula

$$p_c = \frac{1}{\sqrt{q-1}}, \tag{2}$$

proposed by Galam and Mauger [20,21] obtained fair results in two dimensions, but not for higher dimensions.

Galam and Mauger [22,23] provided estimates for several lattices using the power law formula

$$p_c = p_0[(d-1)(q-1)]^{-a} d^b, \tag{3}$$

where the parameters p_0, a, and b were determined by a fit to several exactly known or precisely estimated percolation threshold values. The lattices studied were classifed into three "universality classes." For the two classes of lattices with dimensions $d \leqslant 7$, $b=0$ for site percolation and $b=a$ for bond percolation. One of these classes includes the two-dimensional square, triangular, hexagonal, and dice lattices, for which $p_0=0.8889$ and $a=0.3601$ for site models and $p_0=0.6558$ and $a=0.6897$ for bond models. The other class includes the Kagomé lattice and other lattices with $3 \leqslant d \leqslant 7$, for which $p_0=1.2868$ and $a=0.6160$ for site percolation and $p_0=0.7541$ and $a=0.9346$ for bond percolation. (The third class, which we will not consider here, consists of lattices in eight or more dimensions.) For the lattices considered in [22], the maximum deviation of the formula from numerical estimates is ± 0.008.

Since the universality classes are not precisely defined in terms of properties of the lattice graphs, it is unclear to which class a new lattice belongs. However, in practice, the class for a particular lattice can be assigned if either the bond or site threshold is known: Compute the threshold estimates for both class formulas, then assign the class corresponding to the most accurate estimate. This procedure assigns the same class for both bond and site models for the lattices we consider.

Although the formulas of Galam and Mauger are in extremely good agreement with simulation estimates for the lattices studied, some numerical discrepancies have been noted in the past. Van der Marck [24] noted that if there is to be a universal formula for percolation thresholds, it needs to be based on more information than d and q only. As examples, he provides two three-dimensional lattices with $d=3$ and $q=8$, the body centered cubic lattice, and the stacked triangular lattice. Their site percolation threshold estimates are 0.246 and 0.2623 respectively, with bond percolation estimates of 0.1803 and 0.1859, respectively. Babalievski [25] investigated and confirmed a discrepancy of 0.020 in the estimate for the value of the bond percolation threshold of the ferrovariant of the dodecagonal lattice. However, much larger errors exist: Wierman [26] pointed out that there are errors in the values of these formulas as large as 0.3434.

This study proposes the following collection of desirable properties for universal formulas, as part of the development of a framework for evaluating various universal formulas.

The ideal universal formula for percolation thresholds will

(i) be well-defined;
(ii) be easily computable;
(iii) provide values only between 0 and 1;
(iv depend only on the adjacency structure of the lattice;
(v) be accurate;

(vi) be consistent with the duality relationship (for bond models) and the corresponding matching relationship (for site models);

(vii) be consistent with the containment principle;

(viii) be consistent with the contraction principle (for bond models);

(ix) be consistent with subdivision of edges (for bond models).

The first three properties are necessary for any reasonable formula. Accuracy of predictions is the single most important property. The final four ask that certain theoretical properties that have been proved for percolation thresholds hold for the predictive formulas.

In Sec. II, we explain the meaning and justification of each property. However, we also devote a separate section, Sec. III, to a discussion of accuracy, due to its importance. The results of our evaluation are summarized in Sec. IV.

Sections II and III also contain evaluations of the four universal formulas that were mentioned above: the formula of Vyssotsky-Gordon-Frisch-Hammersley (VGFH) and the Galam-Mauger power law (GM-pl) for bond thresholds, and the Galam-Mauger square root formula (GM-sr) and Galam-Mauger power law (GM-pl) for site thresholds. In this initial investigation, we limit ourselves to formulas for two-dimensional infinite graphs, since there is only imprecise knowledge of percolation threshold values in other dimensions. These four formulas were chosen as the most cited in the literature. All four universal formulas studied here are based on the average degree of the infinite graph. For the average degree to be well-defined, and for appropriateness for applications, we restrict consideration to the prediction of percolation thresholds for infinite two-dimensional periodic graphs, where a periodic graph is as defined by Kesten ([4], pp. 10–11): An infinite graph is *periodic* in d dimensions if it is connected, locally finite, contains no loops, and may be embedded in $\mathbb{R}^d$ in such a way that each coordinate vector is a period for the image and every compact set of $\mathbb{R}^d$ intersects only finitely many edges. Furthermore, we consider only graphs that have no pendant vertices, i.e., vertices of degree 1, since such vertices cannot contribute to percolative behavior.

II. DISCUSSION OF PROPERTIES

We now elaborate on each of the desirable properties for percolation thresholds listed in our evaluation framework.

A. Well-defined

A universal formula should give a well-defined unique value for every periodic graph. Due to periodicity, the average degree can be defined naturally as the limit of the average degrees for a sequence of finite rectangles expanding in both dimensions, or equivalently as the average degree of the graph in a rectangle of one period width in each dimension. Thus, the VGFH and GM-sr formulas are well-defined. For the GM-pl formulas, as noted in the Introduction, if either the bond or site threshold is known for a lattice, a universality class can be assigned. However, the GM-pl formulas do not provide a clear definition of the "universality classes." Since it is not clear which formula applies to any particular graph, the GM-pl formulas do not satisfy this property.

B. Easily computable

A universal formula should be a function of graph parameters that are easily computed. The formulas considered in this initial investigation are all functions of the average degree, which is relatively easily determined, as long as the fundamental periodic region contains few vertices.

However, estimates based on a limit of the number of spanning trees in a region (as the region expands) have been proposed [27–29], and in early percolation studies connections with the connective constant of self-avoiding walks on the lattice were considered. Both of these values are exactly known for few lattices, and reasonably accurate estimates of the unknown values may take substantial computational effort. Dependence upon such parameters seriously limits the usefulness of the universal formula.

C. Values in [0,1]

Since the percolation threshold is a probability value, its value for any lattice graph must lie in the interval $[0,1]$. One would expect this to be trivially satisfied for any proposed universal formula, and it is for the VGFH and GM-sr universal formulas. However, it fails dramatically for GM-pl formulas as the average degree approaches two: For bond thresholds, the limit as $q \to 2$ is $p_0 2^a \approx 1.0578$ for the class 1 formula and is 1.4414 for the class 2 formula. For site thresholds, the limit for the class 2 formula is $p_0 = 1.2868$.

Note that the limiting values are not actually attained by a lattice with $q=2$. However, lattice graphs with q slightly larger than 2 can be constructed. For example, consider inserting a diagonal path with n vertices in each face of the square lattice. By letting n increase, q may be made arbitrarily close to 2.

D. Adjacency structure

From the definition of the percolation model, the percolation threshold depends only on the adjacency structure of the lattice. All four universal formulas investigated here are functions of the average degree, and thus satisfy this property. However, there is a proposal in the literature to estimate percolation thresholds on the basis of a filling factor [30], i.e., the portion of the plane that is covered by certain disks centered at the vertices. A universal formula based on a filling factor would depend on the embedding of the lattice into the plane, which is irrelevant for the percolation model and thus the value of the percolation threshold of the lattice.

E. Accuracy

There are several difficulties in assessing the accuracy of universal formulas. One cannot determine the precise error that a formula makes except in the few cases that the critical probability is exactly known. Even when the precise error is known for a set of lattices, there are options of comparing different universal formulas on the basis of maximum error,

median error, or mean error. Of course, these values will be dependent upon the set of lattice graphs used for the comparison. In the case of the Galam and Mauger power law estimates, there is the additional complication that the universality class of a lattice is not well-defined. We discuss some approaches for dealing with these issues, and carry out an analysis of accuracy, in Sec. III.

F. Duality and matching

An important theorem in percolation theory, due to Kesten [4], establishes the relationship of bond percolation thresholds for a pair of dual planar graphs, G and G^*. If the graphs have an axis of symmetry, then

$$p_c(G) + p_c(G^*) = 1.$$

Thus, it is desirable for a universal formula to provide predictions of the bond percolation thresholds $\hat{p}_c(G)$ and $\hat{p}_c(G^*)$ which satisfy

$$\hat{p}_c(G) + \hat{p}_c(G^*) = 1.$$

For site percolation, Kesten's theorem proves that the same relationship holds for percolation thresholds of pairs of matching graphs, which were introduced by Sykes and Essam [31]. While we will not give the rather complicated definition here, note that the line graphs of a pair of dual planar graphs are a pair of matching graphs. It is also desirable that a universal formula for site percolation thresholds be consistent with the matching relationship.

A graph that is isomorphic to its dual graph is called *self-dual*, and similarly we may define *self-matching*. As examples, the square lattice is self-dual and the triangular lattice is self-matching. Kesten's results imply in both cases that the appropriate percolation threshold is equal to one-half. Note that if a universal formula fails to be consistent with the duality (or matching) relationship, it is possible that it is still consistent with self-duality (or self-matching).

Three of the formulas under investigation here fail both the duality or matching relationship and the self-duality or self-matching property, since the estimate for the bond percolation threshold for the square lattice and the site percolation threshold for the triangular lattice are not one-half. The GM-sr formula is particularly poor with an estimate of 0.4472 for the triangular lattice site threshold. The GM-pl formulas have relatively small errors regarding self-duality and self-matching, but have larger errors when considering dual or matching pairs.

The VFGH formula is perfectly consistent with duality, and thus also self-duality. This was shown by Sykes and Essam [31] using Euler's formula together with the one-to-one correspondences between edges in the graph and its dual and between faces and vertices of the two graphs.

G. Containment

Fisher's containment principle [32] states that if G is a subgraph of H, then

$$p_c(G) \geqslant p_c(H)$$

for both bond and site models. Thus, a universal formula should provide estimates satisfying

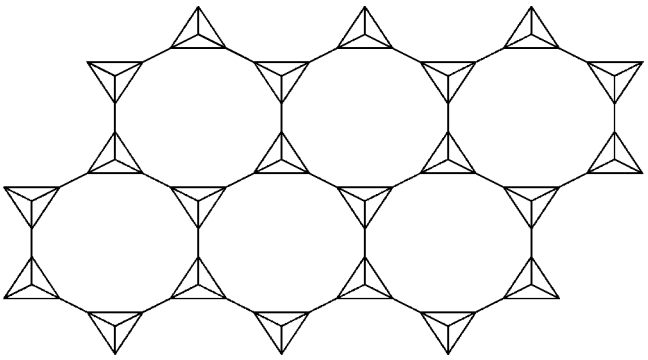

FIG. 1. Contraction example. Contract all six edges in each of the triangles. The original graph has average degree 3.75, while the contracted graph is the hexagonal lattice, which has average degree 3.

$$\hat{p}_c(G) \geqslant \hat{p}_c(H)$$

as well.

We now show that the VGFH and GM-sr universal formulas are not consistent with the containment principle, since they are monotone nonincreasing functions of the average degree: If we take a periodic graph G with average degree greater than 2, we can add sufficiently long paths periodically to obtain a graph H with smaller average degree than G. Then, G is a subgraph of H, so $p_c(G) \geqslant p_c(H)$ by the containment principle, but $q(G) > q(H)$, so the formulas provide estimates $\hat{p}_c(G) < \hat{p}_c(H)$.

The analysis is complicated because the GM-pl formulas are not well-defined. The argument above is valid for the formula within each class. However, it is possible that there is a graph G which is a subgraph of H where the two graphs are in different classes, so that different formulas apply. Without a specification of the GM classes, we have not constructed an example to show that the GM-pl formulas are inconsistent with the containment principle.

H. Contraction

The contraction principle, introduced by Wierman [33], states that if G is obtained by contracting edges in H, then

$$p_c(G) \leqslant p_c(H)$$

for bond models. Thus, a universal formula should provide estimates satisfying

$$\hat{p}_c(G) \leqslant \hat{p}_c(H)$$

as well.

We now show that the VGFH and GM-sr universal formulas are not consistent with the contraction principle, since they are monotone nonincreasing functions of the average degree. In Fig. 1, we show a graph with average degree 3.75 in which edges can be contracted to obtain a graph (the hexagonal lattice) with average degree 3. Thus, the estimated value for the threshold of the contraction graph is less than the estimate for the threshold of the original graph.

The fact that the GM-pl formulas are not well-defined affects this analysis in the same way as for containment.

TABLE I. Numerical comparison of the VGFH and GM-pl bond percolation formulas for nine different periodic lattices which are exactly or nearly exactly solved. Each superscript in the column labeled GM-pl value indicates which class formula provides the value given.

Lattice name	Value or bounds	q	VGFH value	VGFH error	GM-pl value	GM-pl error
$(3, 12^2)$	0.7395,0.7415	3	0.6667	0.0728	0.7541^2	0.0126
Hexagonal	0.6527	3	0.6667	0.0140	0.6558^1	0.0031
D(Bowtie)	0.5955	10/3	0.6000	0.0045	0.5897^1	0.0058
Kagomé	0.5216,0.5277	4	0.5000	0.0216	0.5162^2	0.0054
Square	0.5000	4	0.5000	0.0000	0.4958^1	0.0042
Dice	0.4723,0.4784	4	0.5000	0.0216	0.4958^1	0.0174
Bowtie	0.4045	5	0.4000	0.0045	0.4066^1	0.0021
Triangular	0.3473	6	0.3333	0.0140	0.3486^1	0.0013
$D(3,12^2)$	0.2585,0.2605	6	0.3333	0.0728	0.3203^2	0.0598

I. Subdivision

Given a graph G, let G_k denote the graph obtained by subdividing each edge of G into k edges, i.e. by replacing each edge of G by a path of k edges. For bond percolation

$$p_c(G_k) = [p_c(G)]^{1/k},$$

so it is desirable that a universal formula provide estimates satisfying

$$\hat{p}_c(G_k) = [\hat{p}_c(G)]^{1/k}.$$

Consider the subgraph of G contained in a rectangular region, containing n vertices and e edges and thus average degree $q=2e/n$. Then G_k contains $n^*=n+(k-1)e$ vertices and $e^*=ke$ edges, so

$$q(G_k) = \frac{2e^*}{n^*} = \frac{2kq}{2+(k-1)q}.$$

Taking limits as the rectangular region expands provides the relationship for the average degrees of the infinite graphs.

To investigate consistency of the VGFH formula with subdivision, we compare $[\hat{p}_c(G)]^{1/k}$ with

$$\hat{p}_c(G_k) = \frac{2/q + k - 1}{k}.$$

There are large discrepancies between the two formulas, since as $q\to\infty$, $[\hat{p}_c(G)]^{1/k}=(2/q)^{1/k}$ converges to zero, while $\hat{p}_c(G_k)$ converges to $(k-1)/k$.

For the GM-pl formula, we compare

$$[\hat{p}_c(G)]^{1/k} = p_0\left(\frac{2}{(q-1)}\right)^{a/k}$$

with

$$\hat{p}_c(G_k) = p_0\left(\frac{4(k-1)/q+2}{(2k-1-2(k-1)/q)}\right)^{a}.$$

Again, as $q\to\infty$, the first quantity converges to zero, while the second tends a positive constant.

Therefore, none of the VGFH or GM-pl universal formulas provide an adequate approximation under subdivision.

III. ACCURACY

Our evaluation of the accuracy of the universal formulas has two components. (1) For a selection of graphs, we determine the errors made by each formula, and consider the maximum, median, and mean error. (2) We consider theoretical evidence regarding the maximum errors made by the formulas.

Throughout this discussion, if a lattice graph has been identified by Galam and Mauger as being in a specific class, we used only the formula value for that class. For other lattices, we computed the formula values for both classes of low-dimensional lattices, and computed errors using the formula closest to the exact value, bounds, or estimates used as the "correct" value. (The Galam-Mauger class used for each lattice is indicated by a superscript in Tables I and III.) We recognize that this procedure gives an advantage (perhaps significant) to the Galam and Mauger power law formulas.

A. Accuracy of bond model formulas

For the evaluation of the bond percolation threshold formulas, we selected nine lattice graphs. The exact critical probability value is known for five of the graphs—the square [3], triangular and hexagonal [5], and bowtie and bowtie dual lattices [6]: The other four graphs are the only lattices in the physical science literature for which the bond percolation thresholds are nearly exactly known [16–18], i.e., bounded in an interval of length less than 0.01.

Table I provides the numerical comparisons of the VGFH and GM-pl bond threshold formulas. The maximum error is smaller for the GM-pl formula than for the VGFH formula: 0.0598 versus 0.0728. The median error for the VGFH formula (0.0140) is much larger than that of the GM-pl formula (0.0054). The average error of the VGFH formula (0.0251) is more than twice as large as the GM-pl formula (0.0124), mainly since two errors are quite large rather than just one. Note that even allowing the Galam and Mauger estimate to use the formula for the class that gives the closest result, the VGFH formula still has a smaller error for two of the nine lattices. These comparisons are summarized in Table II.

Wierman [34] has shown that there exist graphs with average degree 6 that have bond percolation thresholds arbi-

TABLE II. Comparison of bond percolation formula accuracy of the VGFH and GM-pl formulas by three measures of error.

	VGFH	GM-pl
Maximum	0.0728	0.0598
Median	0.0140	0.0054
Mean	0.0251	0.0124

TABLE IV. Comparison of site percolation formula accuracy of the GM-sr and GM-pl formulas by four measures of error.

	GM-sr	GM-pl
Maximum	0.1008	0.0552
Median	0.0502	0.0048
Mean	0.0485	0.0179

trarily close to zero, and by duality, graphs with average degree 3 that have bond percolation thresholds arbitrarily close to 1. Thus, the VGFH formula makes errors of at least 0.3333 and the GM-pl formula makes errors of at least 0.3206.

B. Accuracy of site model formulas

There are only three lattices in the physical science literature for which the exact site percolation threshold is known, and there are no nearly exactly solved cases. For this reason, we must use simulation estimates as a standard for evaluating site threshold formulas. We will rely on high precision simulation estimates of the site percolation thresholds of the 11 Archimedean lattices by Suding and Ziff [8].

Table III provides the numerical comparisons of the GM-sr and GM-pl site threshold formulas. The GM-sr formula underestimates the percolation threshold for 10 of the 11 graphs. It also has nearly twice as large a maximum error (0.1008) as the GM-pl formula (0.0552). The median errors are strikingly different: 0.0502 for GM-sr versus. 0.0048 for GM-pl. The average error of the GM-sr formula (0.0485) is almost three times larger than that of the GM-pl formula (0.0179). These error measures are summarized in Table IV.

IV. EVALUATION OF UNIVERSAL FORMULAS

We summarize our evaluation of the four universal formulas in Table V. For each of the nine desirable properties, we give our evaluation.

Some items for the GM-pl formulas are marked "Partly," to indicate that the answer depends on the specification of classes of graphs used to make the formula well-defined. The GM-pl formulas are computable only if the class can be determined. With information about the bond or site threshold, a class can be assigned to a lattice, and the GM-pl value can be easily computed. The adjacency property will be satisfied if the definition of classes depends only on the adjacency structure. Consistency with the containment and contraction principles depends on whether or not the related graphs can be of different classes. As discussed in Sec. III, the accuracy depends heavily on the class definition, since the formula values for the two classes may differ by as much as 0.1470 for site models and as much as 0.0983 for bond models.

The rating for duality and matching are to be interpreted as follows. "Yes" indicates that the VGFH formula has been proved to be consistent with duality. "Fair" indicates that the formula is not consistent with either duality or self-duality (for bond models) or with either matching or self-matching (for site models), but the difference from one of the critical probability sums of appropriate pairs of graphs is relatively small. "Poor" is not consistent, as for "Fair," but with relatively large numerical errors. Although not used here, "Good" would indicate that the formula is not consistent with duality, but is consistent with self-duality, for example.

To evaluate accuracy, we have set absolute standards for each of the three summary statistics for the error—maximum, mean, and median—for ratings of Excellent, Good, Fair, and Poor. The standards for the mean and median are the same, while those for the maximum are twice as high.

TABLE III. Numerical comparison of site percolation formula values for the 11 Archimedean lattices, using estimates by Suding and Ziff. Each superscript in the column labeled GM-pl value indicates which class formula provides the value given.

Lattice name	Suding-Ziff estimate	q	GM-sr value	GM-sr error	GM-pl value	GM-pl error
$(3, 12^2)$	0.8079…	3	0.7071	0.1008	0.8396^2	0.0317
(4, 6, 12)	0.7478	3	0.7071	0.0307	0.6926^1	0.0552
$(4, 8^2)$	0.7297	3	0.7071	0.0226	0.6926^1	0.0371
Hexagonal	0.6970	3	0.7071	0.0101	0.6926^1	0.0044
Kagomé	0.6527…	4	0.5774	0.0753	0.6540^2	0.0013
(3, 4, 6, 4)	0.6218	4	0.5774	0.0444	0.5985^1	0.0233
Square	0.5937	4	0.5774	0.0163	0.5985^1	0.0048
$(3^4, 6)$	0.5795	5	0.5000	0.0795	0.5478^2	0.0317
$(3^2, 4, 3, 4)$	0.5508	5	0.5000	0.0508	0.5478^2	0.0030
$(3^3, 4^2)$	0.5502	5	0.5000	0.0502	0.5478^2	0.0024
Triangular	0.5000…	6	0.4472	0.0528	0.4979^1	0.0021

TABLE V. Summary of evaluations of universal formulas.

	Bond model		Site model	
Property	VGFH	GM-pl	GM-sr	GM-pl
Well-defined	Yes	No	Yes	No
Computable	Yes	Partly	Yes	Partly
Values in [0,1]	Yes	No	Yes	No
Adjacency	Yes	Partly	Yes	Partly
Accuracy (maximum)	Fair	Fair	Poor	Fair
Accuracy (mean)	Good	Excellent	Poor	Excellent
Accuracy (median)	Fair	Good	Fair	Good
Duality and matching	Yes	Fair	Poor	Fair
Containment	No	No	No	No
Contraction	No	No	N.A.	N.A.
Subdivision	No	No	N.A.	N.A.

Admittedly, the levels are somewhat arbitrary, but we believe that they make useful distinctions. The rating standards are summarized in Table VI. We have opted not to combine these ratings into a single measure of accuracy, so all three ratings are included in Table V.

The evaluation shows that there is considerable room for improvement in universal formulas for percolation thresholds. All the formulas studied fail at least three of the nine criteria. Although much older, the VGFH satisfies more of the properties than the GM-pl formula for bond thresholds. Between the two site threshold formulas, the GM-pl formula fails more properties but is much more accurate than the GM-sr formula.

V. FUTURE RESEARCH DIRECTIONS

The work described in this article is a preliminary investigation of a few universal formulas. There are three related directions, described briefly in the following subsections, for future research that are particularly interesting to the authors.

A. Other formulas or methods

Researchers have considered other means of developing universal formulas for the percolation threshold, based on a minimal spanning tree approach [27–29], lattice Green functions [35], filling factor [30], and preferred directions for cluster formation [36], which can be evaluated using the framework of this paper.

TABLE VI. Standards for ratings of accuracy.

	Maximum	Median	Mean
Poor	Above 0.1000	Above 0.0500	Above 0.0500
Fair	0.0500 to 0.1000	0.0250 to 0.0500	0.0250 to 0.0500
Good	0.0200 to 0.0500	0.0100 to 0.0250	0.0100 to 0.0250
Excellent	Below 0.0200	Below 0.0100	Below 0.0100

B. Relationship between formulas for bond and site models

In [23], Galam and Mauger extended their formula via the use of an effective parameter q_{eff} to replace the average coordination number q. They suggest that their formula has predicting ability for percolation thresholds which have not yet been computed. For example, if the site threshold of a lattice has been estimated, q_{eff} can be computed from the formula for site thresholds, and can be used to predict the bond threshold from the formula for bond thresholds.

The Galam & Mauger extension raises the issue of evaluating the relationship between formulas for the bond threshold and the site threshold. We have already identified some desirable properties.

(1) The universal formulas should be consistent with the bond-to-site transformation, satisfying

$$\hat{p}_c(G \text{ bond}) = \hat{p}_c(L(G)\text{site}),$$

where $L(G)$ denotes the covering graph (also called the *line graph* in the mathematical literature) of G. This equality holds for percolation thresholds by an early observation of Fisher [32].

(2) For every graph, the universal formulas should satisfy $\hat{p}_c(G \text{ bond}) \leqslant \hat{p}_c(G \text{ site})$. Hammersley [37] proved that this holds for percolation thresholds of every infinite graph.

(3) The universal formulas should not imply that two graphs must have their bond percolation and site percolation thresholds in the same order. While this was commonly believed for many years, Wierman [38] provided examples of pairs of graphs with the site percolation thresholds in the opposite order to their bond percolation thresholds.

C. Developing improved universal formulas

The ultimate goal of this research program is to develop improved universal formulas. This is a challenging problem, since it is clear from our analysis that the existing universal formulas leave much to be desired.

A way must be found to incorporate other information besides the dimension and the average degree of the lattice. This can be seen in Table III, since graphs with the same degree have different percolation thresholds. For example, the four Archimedean lattices which are regular with degree 3 have estimated site percolation thresholds ranging from 0.6970 to 0.8079.

The major challenges are to identify other features of the graphs that play a role in determining the percolation threshold, and to formulate improved universal formulas which satisfy more of the desirable properties than the existing universal formulas.

As one possibility, Wierman and Vahidi [39] suggested that the variability of the vertex degree has an effect on the percolation threshold, with higher variability leading to lower critical probability values. For planar lattice bond percolation models, the variability of degree of the dual lattice corresponds to the variability of the number of sides of faces in the original lattice, so characteristics of the faces may play a role. Formulation of these ideas will necessarily involve

investigation of alternative measures of variability of degree and numbers of edges of faces.

A possible approach is to employ the existing formulas in developing improved formulas. For example, since the VGFH formula satisfies more properties than the others, we plan to study the conversion of the VGFH formula via the bond-to-site transformation to derive a site percolation threshold formula which might outperform the GM-sr and GM-pl formulas.

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[1] S. R. Broadbent and J. M. Hammersley, Proc. Cambridge Philos. Soc. **53**, 629 (1957).
[2] R. Lyons, Ann. Prob. **18**, 931 (1990).
[3] H. Kesten, Commun. Math. Phys. **74**, 41 (1980).
[4] H. Kesten, *Percolation Theory for Mathematicians* (Birkhäuser, Boston, 1982).
[5] J. C. Wierman, Adv. Appl. Probab. **13**, 298 (1981).
[6] J. C. Wierman, J. Phys. A **17**, 1525 (1984).
[7] D. Stauffer and A. Aharony, *Introduction to Percolation Theory* (Taylor & Francis, London, 1991).
[8] P. N. Suding and R. M. Ziff, Phys. Rev. E **60**, 275 (1999).
[9] J. van den Berg and A. Ermakov, Random Struct. Algorithms **8**, 199 (1996).
[10] T. Łuczak and J. C. Wierman, J. Phys. A **21**, 3131 (1988)
[11] J. C. Wierman, in *Disorder in Physical Systems*, edited by G. Grimmett and D. J. A. Welsh (Oxford University Press, Oxford, 1990), pp. 349–360.
[12] J. C. Wierman, Combinatorics, Probab. Comput. **4**, 181 (1995).
[13] J. C. Wierman, Congr. Numer. **150**, 117 (2001).
[14] J. C. Wierman, Random Struct. Algorithms **20**, 507 (2002).
[15] J. C. Wierman, Combinatorics, Probab. Comput. **11**, 629 (2002).
[16] J. C. Wierman, Combinatorics, Probab. Comput. **12**, 95 (2002).
[17] W. D. May and J. C. Wierman, Combinatorics, Probab. Comput. (to be published).
[18] W. D. May and J. C. Wierman, Congr. Numer. **162**, 5 (2003).
[19] V. A. Vyssotsky, S. B. Gordon, H. L. Frisch, and J. M. Hammersley, Phys. Rev. **123**, 1566 (1961).
[20] S. Galam and A. Mauger, J. Appl. Phys. **75**, 5526 (1994).
[21] S. Galam and A. Mauger, Physica A **205**, 502 (1994).
[22] S. Galam and A. Mauger, Phys. Rev. E **53**, 2177 (1996).
[23] S. Galam and A. Mauger, Phys. Rev. E **56**, 322 (1997).
[24] S. C. van der Marck, Phys. Rev. E **55**, 1228 (1997).
[25] F. Babalievski, Phys. Rev. E **59**, 1278 (1999).
[26] J. C. Wierman, Phys. Rev. E **66**, 027105 (2002)
[27] C. Dussert, G. Rasigni, M. Rasigni, J. Palmari, and A. Llebaria, Phys. Rev. B **34**, 3528 (1986).
[28] C. Dussert, G. Rasigni, and M. Rasigni, Phys. Lett. A **139**, 35 (1989).
[29] C. d'Iribarne, G. Rasigni, and M. Rasigni, Phys. Lett. A **209**, 95 (1995).
[30] H. Scher and R. Zallen, J. Chem. Phys. **53**, 3759 (1970).
[31] M. F. Sykes and J. W. Essam, J. Math. Phys. **5**, 1117 (1964).
[32] M. E. Fisher, J. Math. Phys. **2**, 620 (1961).
[33] J. C. Wierman, J. Phys. A **21**, 1487 (1988).
[34] J. C. Wierman, J. Phys. A **35**, 959 (2002).
[35] M. Sahimi, B. D. Hughes, L. E. Scriven, and H. T. Davis, J. Phys. A **16**, L67 (1983).
[36] A. Rosowsky, Eur. Phys. J. B **15**, 77 (2000).
[37] J. M. Hammersley, J. Math. Phys. **2**, 728 (1961).
[38] J. C. Wierman, Discrete Appl. Math. **129**, 545 (2003).
[39] J. C. Wierman and M. Q. Vahidi-Asl, Congr. Numer. **154**, 201 (2002).